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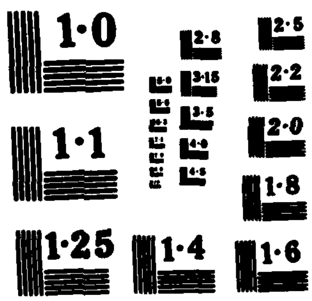
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Technical Report #84-33

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LOCALLY OPTIMAL SUBSET SELECTION RULES BASED ON RANKS UNDER JOINT TYPE II CENSORING

1. Introduction

Let $\pi_1, \ldots, \pi_k$ be k ($\geq$ 2) independent populations where π_i has the associated distribution function $F(x, \theta_i)$ and density $f(x, \theta_i)$ with the unknown parameter θ_i belonging to an interval (a, b) of the real line. Our goal is to select a subset (preferably small in size) of the k populations $\pi_1, \ldots, \pi_k$ that will contain the best (suitably defined) among them.

In practice, it sometimes happens that the actual values of the random variables can only be observed under great cost, or not at all, while their ordering is readily observable. This occurs for instance in life-testing when one only observes the order in which the parts under investigation fail without being able to record the actual time of failure. In problems of this type, one may desire to investigate decision rules based on ranks.

In dealing with the goal specified above, Gupta and McDonald (1970) studied three classes of subset selection rules based on ranks for selecting a subset containing the best among k populations when the underlying distributions are unknown. When the form of the underlying distribution is known but the values of the parameters θ_i, $i = 1, \ldots, k$, are unknown, Gupta, Huang and Nagel (1979) studied some locally optimal subset selection rules based on ranks. The latter study leads to the conclusion that the class of subset selection rules R_3 of Gupta and McDonald (1970) is locally optimal in some sense. Huang and Panchapakesan (1982) also studied the problem of deriving some subset selection rules, based on ranks, which are locally optimal in the sense that the rules have the property of strong monotonicity. All the studies mentioned above only considered the situation where the ranks are completely observed.

We now consider a problem as follows: Suppose that there are k different devices and we want to select the best among them. From each kind of device, say π_i, n prototypes are taken for experiment and the N = kn prototypes are simultaneously put on a life test. Due to design reasoning or cost consideration, the experiment terminates as soon as the first r failures among the N devices are observed for some predetermined value r, where $1 \leq r \leq N$. Based on these r observations, we want to ascertain which device is associated with the largest (expected) lifetime. Since we are only concerned with the first r failures, we call this censoring scheme as a joint type II censoring.

In this paper, we are interested in deriving subset selection rules which satisfy the basic P^*-condition and locally maximize the probability of a correct selection among all invariant subset selection rules based on the ranks under the joint type II censoring. We assume that the functional form of the density function $f(x,\theta)$ is known but the value of the parameter θ is unknown. In Section 2, the problem is formulated. Some properties related to the ranks under the joint type II censoring are also given. Following the earlier setup of Gupta, Huang and Nagel (1979), a locally optimal subset selection rule R_1 is derived in Section 3. The property of local monotonicity related to the rule R_1 is also discussed in Section 4. Finally, a comparison between the subset selection rule R_1 and that of Huang and Panchapakesan (1982) is discussed in Section 5.

2. Formulation of the Problem

Let $\pi_1, \ldots, \pi_k$ be k (≥ 2) populations and let $f(x, \theta_i)$ be the density function associated with the population π_i for $i = 1, \ldots, k$. Let $\theta_{[1]} \leq \cdots \leq \theta_{[k]}$ be the ordered parameters of $\theta_1, \ldots, \theta_k$. Of course, the correct pairing of the ordered and unordered θ_i is unknown to us. The population associated with $\theta_{[k]}$ is called the best population. In case of a tie, one of the contenders is tagged and is called the best. Let $\Omega = \{\underset{\sim}{\theta} | \underset{\sim}{\theta} = (\theta_1, \ldots, \theta_k)\}$ and $\Omega_0 = \{\underset{\sim}{\theta} \in \Omega | \theta_1 = \cdots = \theta_k\}$. Let X_{ij}, $j = 1, \ldots, n$ be independent observations from π_i and let R_{ij} denote the rank of X_{ij} in the pooled sample of the $N = kn$ observations. The smallest observation has rank 1 and the largest has rank N. Let $x_1 \leq \cdots \leq x_N$ denote the ordered observations.

Definition 2.1. A rank configuration is an N-tuple $\underset{\sim}{\Delta} = (\Delta_1, \ldots, \Delta_N)$, $\Delta_i \in \{1, \ldots, k\}$ where $\Delta_i = j$ means that the ith smallest observation in the pooled sample comes from π_j.

Let $L = \{\underset{\sim}{\Delta}\}$ denote the set of all rank configurations. For each $\underset{\sim}{\Delta} \in L$, let $\chi_{\underset{\sim}{\Delta}} = \{\underset{\sim}{x} = (x_1, \ldots, x_N) \in \chi \mid \underset{\sim}{\Delta}_{\underset{\sim}{x}} = \underset{\sim}{\Delta}\}$, where $\chi = \{\underset{\sim}{x} \mid \underset{\sim}{x} = (x_1, \ldots, x_N)\}$ and $\underset{\sim}{\Delta}_{\underset{\sim}{x}}$ denotes the rank configuration of $\underset{\sim}{x} = (x_1, \ldots, x_N)$.

Let r be a predetermined integer such that $1 \leq r \leq N$. Under the joint type II censoring scheme, only the first r smallest observations in the pooled sample of the N (X_{ij}, $j = 1, \ldots, n$; $i = 1, \ldots, k$) are observed. That is, for the rank configuration $\underset{\sim}{\Delta}_{\underset{\sim}{x}} = (\Delta_1, \ldots, \Delta_N)$, only the subvector $(\Delta_1, \ldots, \Delta_r)$ is observable. For this preassigned value r, let C_r be a function defined on L such that for each $\underset{\sim}{\Delta} = (\Delta_1, \ldots, \Delta_N) \in L$, $C_r(\underset{\sim}{\Delta}) = (\Delta_1, \ldots, \Delta_r) = \underset{\sim}{\Delta}(r)$. Let $L_r = C_r(L)$.

Then, $L_r = \{\underset{\sim}{\Delta}(r) \mid \underset{\sim}{\Delta} \in L\}$. Hence, for each $\underset{\sim}{\Delta}(r) \in L_r$, $\max(0, r - (k-1)n) \leq r_i \equiv \sum_{j=1}^{r} I_{\{i\}}(\Delta_j) \leq \min(r, n)$ for each $i = 1, \ldots, k$, and $\sum_{i=1}^{k} r_i = r$. We call $\underset{\sim}{\Delta}(r)$ as a joint type II censored rank configuration.

For each $\underset{\sim}{\Delta}(r) \in L_r$, define the set $L(\underset{\sim}{\Delta}(r)) = \{\underset{\sim}{\Delta} \in L \mid C_r(\underset{\sim}{\Delta}) = \underset{\sim}{\Delta}(r)\}$. Let $|A|$ denote the number of elements in the set A. Then,

$$|L(\underset{\sim}{\Delta}(r))| = \prod_{m=1}^{k} \binom{N - r - \sum_{i=1}^{m-1} (n - r_i)}{n - r_m} .$$

where $\sum_{i=1}^{0} \equiv 0$. Also,

$$\sum_{\underset{\sim}{\Delta}(r)\in\mathcal{L}_r} |\mathcal{L}(\underset{\sim}{\Delta}(r))| = N!/(n!)^k$$

Let $\mathcal{D}$ be the decision space consisting of all the 2^k subsets of the set $\{1, \ldots, k\}$. Any subset is denoted by d so that $\mathcal{D} = \{d | d \subset \{1, \ldots, k\}\}$. A decision d is the selection of a subset of the k populations. The fact that $i \in d$ means that population π_i is included in the selected subset if decision d is made. Let $\delta(\underset{\sim}{\Delta}(r), d)$ denote the probability that the decision d is made if the censored rank configuration $\underset{\sim}{\Delta}(r)$ is observed. Let $\alpha_i(\underset{\sim}{\Delta}(r))$, $i = 1, \ldots, k$, denote the individual selection probability of the k populations, where

$$\alpha_i(\underset{\sim}{\Delta}(r)) = \sum_{d \ni i} \delta(\underset{\sim}{\Delta}(r), d), \tag{2.1}$$

the summation being over all the subsets containing i.

<u>Definition 2.2</u>. A subset selection rule R based on the censored ranks is a measurable mapping from $\mathcal{L}_r$ into $[0, 1]^k$ such that

$$R(\underset{\sim}{\Delta}(r)) = (\alpha_1(\underset{\sim}{\Delta}(r)), \ldots, \alpha_k(\underset{\sim}{\Delta}(r))).$$

Let $P_i(\underset{\sim}{\theta})$ denote the probability of including the population π_i in the selected subset when $\underset{\sim}{\theta} = (\theta_1, \ldots, \theta_k)$ are the true parameters. That is, $P_i(\underset{\sim}{\theta}) = E_{\underset{\sim}{\theta}}[\alpha_i(\underset{\sim}{\Delta}(r))]$ where the expectation is over the set $\mathcal{L}_r$. Any decision d that corresponds to the selection of the best population is called a correct selection (CS). The probability of a correct selection is denoted by $P_{\underset{\sim}{\theta}}(CS|R)$ when the subset selection rule R is applied.

Let G denote the group of permutations g of the integers 1,..., k. We write g(1, ..., k) = (g1, ..., gk). Let h denote the inverse of g and define $g(\theta_1, \ldots, \theta_k) = (\theta_{h1}, \ldots, \theta_{hk})$.

For each $\underset{\sim}{\Delta} \in \mathcal{L}$, $\underset{\sim}{\Delta}(r) \in \mathcal{L}_r$, let $\bar{g}$ and $\bar{\bar{g}}$ be defined by $\bar{g}\underset{\sim}{\Delta} = (g\Delta_1, \ldots, g\Delta_N)$ and $\bar{\bar{g}}\underset{\sim}{\Delta}(r) = (g\Delta_1, \ldots, g\Delta_r)$, respectively. Thus, both $\bar{g}$ and $\bar{\bar{g}}$ are induced from g. Let $\bar{G} = \{\bar{g}\}$ and $\bar{\bar{G}} = \{\bar{\bar{g}}\}$. It is easy to see that $\mathcal{C}_r(\bar{g}\underset{\sim}{\Delta}) = \bar{\bar{g}}(\mathcal{C}_r(\underset{\sim}{\Delta}))$. Also, $\underset{\sim}{\Delta} \in \mathcal{L}(\underset{\sim}{\Delta}(r))$ iff $\bar{g}\underset{\sim}{\Delta} \in \mathcal{L}(\bar{\bar{g}}\underset{\sim}{\Delta}(r))$. Hence,

$$|\mathcal{L}(\underset{\sim}{\Delta}(r))| = |\mathcal{L}(\bar{\bar{g}}\underset{\sim}{\Delta}(r))| \tag{2.2}$$

for each $\underset{\sim}{\Delta}(r) \in \mathcal{L}_r$ and for each $\bar{\bar{g}} \in \bar{\bar{G}}$.

<u>Definition 2.3</u>. A subset selection rule R on $\mathcal{L}_r$ is invariant under permutation if and only if $(\alpha_1(\bar{\bar{g}}\underset{\sim}{\Delta}(r)), \ldots, \alpha_k(\bar{\bar{g}}\underset{\sim}{\Delta}(r))) = g(\alpha_1(\underset{\sim}{\Delta}(r)), \ldots, \alpha_k(\underset{\sim}{\Delta}(r)))$ for all $\underset{\sim}{\Delta}(r) \in \mathcal{L}_r$, $g \in G$ and $\bar{\bar{g}}$ induced from g.

Let $f(x, \theta_i)$ be the density function associated with population π_i, with the parameter θ_i belonging to some interval (a, b) of the real line, where $-\infty \le a < b \le \infty$. Let $\Omega = \{\underset{\sim}{\theta} | \underset{\sim}{\theta} = (\theta_1, \ldots, \theta_k)\}$, $\Omega_0 = \{\underset{\sim}{\theta} \in \Omega | \theta_1 = \cdots = \theta_k\}$ and $\Omega_i = \{\underset{\sim}{\theta} \in \Omega | \theta_i \ge \theta_j \text{ for all } j \ne i\}$. Furthermore, let the density $f(x, \theta)$ have the following properties:

Condition A:

$$\begin{cases} \text{(i)} & f(x, \theta) \text{ is absolutely continuous in } \theta \text{ for every } x; \\ \text{(ii)} & \dot{f}(x, \theta) = \frac{\partial}{\partial\theta} f(x, \theta) \text{ exists and is continuous in } \theta \text{ for every } x; \\ \text{(iii)} & \lim_{\theta \to \theta_0} \int_{-\infty}^{\infty} |\dot{f}(x, \theta)| dx = \int_{-\infty}^{\infty} |\dot{f}(x, \theta_0)| dx < \infty \end{cases}$$

holds for every $\theta_0 \varepsilon (a, b)$.

Now, under the assumptions of Condition A, our goal is to derive an invariant subset selection rule R, based on the joint type II censored ranks, such that

$$\begin{cases} \text{(i)} & \inf_{\underset{\sim}{\theta}_0 \in \Omega_0} P_{\underset{\sim}{\theta}_0}(CS|R) = P^* \text{ where } P^* \varepsilon (\frac{1}{k}, 1) \text{ is prespecified;} \\ \text{(ii)} & P_{\underset{\sim}{\theta}}(CS|R) \text{ is as large as possible for all } \underset{\sim}{\theta} \text{ in a neighborhood of } \underset{\sim}{\theta}_0 \varepsilon \Omega_0. \end{cases}$$

Note that for each $\underset{\sim}{\theta}_0 \varepsilon \Omega_0$, $P_{\underset{\sim}{\theta}_0}(CS|R)$ will be interpreted as the probability of selecting a specified population.

3. A Locally Optimal Subset Selection Rule

For each $\underset{\sim}{\theta} \in \Omega$, $\underset{\sim}{\Delta}(r) \in \mathcal{L}_r$, let $P_{\underset{\sim}{\theta}}(\underset{\sim}{\Delta}(r))$ denote the probability that the joint type II censored rank configuration $\underset{\sim}{\Delta}(r)$ is observed under $\underset{\sim}{\theta}$. Also, let $P_{\underset{\sim}{\theta}}(\underset{\sim}{\Delta})$, $\underset{\sim}{\Delta} \in \mathcal{L}$, denote the probability that the complete rank configuration $\underset{\sim}{\Delta}$ is observed under $\underset{\sim}{\theta}$. Then,

$$P_{\underset{\sim}{\theta}}(\underset{\sim}{\Delta}) = (n!)^k \int_{-\infty}^{\infty} \int_{-\infty}^{x_N} \cdots \int_{-\infty}^{x_2} \prod_{j=1}^{N} f(x_j, \theta_{\Delta_j}) dx_1 \cdots dx_N. \qquad (3.1)$$

It is also clear that for each $\underset{\sim}{\Delta}(r) \in \mathcal{L}_r$,

$$P_{\underset{\sim}{\theta}}(\underset{\sim}{\Delta}(r)) = \sum_{\underset{\sim}{\Delta} \in \mathcal{L}(\underset{\sim}{\Delta}(r))} P_{\underset{\sim}{\theta}}(\underset{\sim}{\Delta}). \qquad (3.2)$$

Let $\underset{\sim}{\theta}_0 = (\theta_0, \ldots, \theta_0) \in \Omega_0$, where $\theta_0 \in (a, b)$. By applying a simple algebraic computation, $P_{\underset{\sim}{\theta}}(\underset{\sim}{\Delta})$ can be written as follows:

$$P_{\underset{\sim}{\theta}}(\underset{\sim}{\Delta}) = (n!)^k \left[A_0(\theta_0) + \sum_{i=1}^{k} (\theta_i - \theta_0) A_i(\underset{\sim}{\Delta}, \underset{\sim}{\theta}_0, \underset{\sim}{\theta})\right] \qquad (3.3)$$

where $A_0(\theta_0) = \int_{-\infty}^{\infty} \int_{-\infty}^{x_N} \cdots \int_{-\infty}^{x_2} \prod_{j=1}^{N} f(x_j, \theta_0) dx_1 \cdots dx_N = \frac{1}{N!}$ which is independent of θ_0,

$$A_i(\underset{\sim}{\Delta}, \underset{\sim}{\theta}_0, \underset{\sim}{\theta}) = \sum_{\substack{j=1 \\ \Delta_j = i}}^{N} \int_{-\infty}^{\infty} \int_{-\infty}^{x_N} \cdots \int_{-\infty}^{x_2} q(i, j, \underset{\sim}{\theta}_0, \underset{\sim}{\theta}, \underset{\sim}{x}) dx_1 \cdots dx_N \qquad (3.4)$$

for each $i = 1, \ldots, k$, where $\underset{\sim}{x} = (x_1, \ldots, x_N)$ and

$$q(i,j,\underset{\sim}{\theta}_0,\underset{\sim}{\theta},\underset{\sim}{x}) = \frac{f(x_j,\theta_i)-f(x_j,\theta_0)}{\theta_i - \theta_0} \prod_{m=1}^{j-1} f(x_m,\theta_0) \prod_{m=j+1}^{N} f(x_m,\theta_{\Delta_m}).$$

Here, we define $\prod_{j=1}^{0} \equiv 1$, $\prod_{j=N+1}^{N} \equiv 1$ and $[f(x_j,\theta_i) - f(x_j,\theta_0)]/(\theta_i - \theta_0) = 0$ if $\theta_i = \theta_0$.

Let $\underset{\sim}{\theta}_0 \varepsilon \Omega_0$ and let $||\underset{\sim}{\theta} - \underset{\sim}{\theta}_0|| = \max_{1 \le i \le k} |\theta_i - \theta_0|$.

Thus, if $\underset{\sim}{\theta} = (\theta_1, \ldots, \theta_k)$ is in the neighborhood of $\underset{\sim}{\theta}_0$ with $\theta_i \neq \theta_0$ for all $i = 1, \ldots, k$, then, under the Condition A, following an argument analogous to a theorem (page 71) of Hajek and Sidak (1967), we have

$$\lim_{||\underset{\sim}{\theta}-\underset{\sim}{\theta}_0|| \to 0} A_i(\underset{\sim}{\Delta}, \underset{\sim}{\theta}_0, \underset{\sim}{\theta}) = A_i^*(\underset{\sim}{\Delta}, \underset{\sim}{\theta}_0) = \sum_{\substack{j=1 \\ \Delta_j = i}}^{N} B_j(\underset{\sim}{\theta}_0), \tag{3.5}$$

and

$$|A_i^*(\underset{\sim}{\Delta}, \underset{\sim}{\theta}_0)| \le \sum_{\substack{j=1 \\ \Delta_j = i}}^{N} |B_j(\underset{\sim}{\theta}_0)| < \infty$$

for each $i = 1, \ldots, k$, where

$$B_j(\underset{\sim}{\theta}_0) = \int_{-\infty}^{\infty} \int_{-\infty}^{x_N} \cdots \int_{-\infty}^{x_2} \dot{f}(x_j,\theta_0) \prod_{\substack{m=1 \\ m \neq j}}^{N} f(x_m, \theta_0) dX_1 \cdots dX_N. \tag{3.6}$$

That is, there exists an $\varepsilon > 0$ such that as $0 < ||\underset{\sim}{\theta} - \underset{\sim}{\theta}_0|| < \varepsilon$, $A_i(\underset{\sim}{\Delta}, \underset{\sim}{\theta}_0, \underset{\sim}{\theta})$ is approximately equal to $A_i^*(\underset{\sim}{\Delta}, \underset{\sim}{\theta}_0)$ for each $i = 1, \ldots, k$.

<u>Lemma 3.1</u>. Suppose that the density function $f(x, \theta)$ satisfies the Condition A. For each $\theta_0 \varepsilon (a, b)$, let $V(\theta_0) = \sum_{j=1}^{N} B_j(\underset{\sim}{\theta}_0)$ where $\underset{\sim}{\theta}_0 = (\theta_0, \ldots, \theta_0) \varepsilon \Omega_0$. Then $V(\theta_0) = 0$ for all $\theta_0 \varepsilon (a, b)$.

Proof: Note that for each $\theta_0 \varepsilon (a, b)$,

$$\sum_{j=1}^{N} B_j(\underset{\sim}{\theta}_0) = \sum_{j=1}^{N} \int_{-\infty}^{\infty} \int_{-\infty}^{x_N} \cdots \int_{-\infty}^{x_2} \dot{f}(x_j, \theta_0) \prod_{\substack{m=1 \\ m \neq j}}^{N} f(x_m, \theta_0) dX_1 \cdots dX_N$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{x_N} \cdots \int_{-\infty}^{x_2} \sum_{j=1}^{N} \dot{f}(x_j, \theta_0) \prod_{\substack{m=1 \\ m \neq j}}^{N} f(x_m, \theta_0) dX_1 \cdots dX_N$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{x_N} \cdots \int_{-\infty}^{x_2} \left[\frac{d}{d\theta} \prod_{m=1}^{N} f(x_m, \theta)\right]\Bigg|_{\theta=\theta_0} dX_1 \cdots dX_N$$

$$= \frac{d}{d\theta} \int_{-\infty}^{\infty} \int_{-\infty}^{x_N} \cdots \int_{-\infty}^{x_2} \prod_{m=1}^{N} f(x_m, \theta) dX_1 \cdots dX_N \Bigg|_{\theta=\theta_0},$$

where the last equality is obtained under Condition A.

Therefore $V(\theta_0) = 0$ for all $\theta_0 \varepsilon (a, b)$ since $\int_{-\infty}^{\infty} \int_{-\infty}^{x_N} \cdots \int_{-\infty}^{x_2} \prod_{m=1}^{N} f(x_m, \theta)\, dX_1 \cdots dX_N = \frac{1}{N!}$ which is independent of θ. This completes the proof of this lemma.

Lemma 3.2. Let $\underset{\sim}{\theta} \in \Omega$ and let $P_i(\underset{\sim}{\theta}) = E_{\underset{\sim}{\theta}}[\alpha_i(\underset{\sim}{\Delta}(r))]$ be the probability of including population π_i in the selected subset under $\underset{\sim}{\theta}$ by applying an invariant subset selection rule R. Let $G(i) = \{g \in G | gi = i\}$. Then,

$$P_i(\underset{\sim}{\theta}) = \sum_{\underset{\sim}{\Delta}(r) \in L_r} \left[\frac{(n!)^k}{N!} |L(\underset{\sim}{\Delta}(r))| + \frac{(n!)^k}{(k-1)!} W(\underset{\sim}{\Delta}(r), \underset{\sim}{\theta}, \underset{\sim}{\theta}_0, G(i))\right] \alpha_i(\underset{\sim}{\Delta}(r))$$

where

$$W(\underset{\sim}{\Delta}(r), \underset{\sim}{\theta}, \underset{\sim}{\theta}_0, G(i)) = \sum_{\underset{\sim}{\Delta} \in L(\underset{\sim}{\Delta}(r))} \sum_{g \in G(i)} \sum_{j=1}^{k} (\theta_{hj} - \theta_0) A_j(\underset{\sim}{\Delta}, \underset{\sim}{\theta}_0, g\underset{\sim}{\theta}),$$

h is the inverse of $g \in G(i)$ and $\underset{\sim}{\theta}_0 = (\theta_0, \ldots, \theta_0) \in \Omega_0$.

Proof: This lemma can be verified by following an argument analogous to that of Gupta, Huang and Nagel (1979, page 257). We omit the detail here.

Lemma 3.3. Suppose that the density function $f(x, \theta)$ satisfies the requirements of Condition A. Let $G(i) = \{g \in G | gi = i\}$. Then,

$$\sum_{g \in G(i)} \sum_{j=1}^{k} (\theta_{hj} - \theta_0) A_j^*(\underset{\sim}{\Delta}, \underset{\sim}{\theta}_0) = (k-2)!(k\theta_i - U) A_i^*(\underset{\sim}{\Delta}, \underset{\sim}{\theta}_0)$$

for each $i = 1, \ldots, k$, for each $\underset{\sim}{\theta} \in \Omega$, $\underset{\sim}{\theta}_0 \in \Omega_0$ where $U = \sum_{j=1}^{k} \theta_j$ and h is the inverse of $g \in G(i)$ and $\underset{\sim}{\Delta} \in L$.

Proof: First note that $\sum_{j=1}^{k} A_j^*(\underset{\sim}{\Delta}, \underset{\sim}{\theta}_0) = \sum_{j=1}^{k} \sum_{\substack{m=1 \\ \Delta_m = j}}^{N} B_m(\underset{\sim}{\theta}_0) = \sum_{m=1}^{N} B_m(\underset{\sim}{\theta}_0) = 0$

which is obtained from Lemma 3.1. Now,

$$\sum_{g\epsilon G(i)} \sum_{j=1}^{k} (\theta_{hj} - \theta_0) A_j^*(\underset{\sim}{\Delta}, \underset{\sim}{\theta}_0)$$

$$= \sum_{g\epsilon G(i)} \sum_{j=1}^{k} \theta_{hj} A_j^*(\underset{\sim}{\Delta}, \underset{\sim}{\theta}_0)$$

$$= \sum_{\substack{j=1 \\ j\neq i}}^{k} A_j^*(\underset{\sim}{\Delta}, \underset{\sim}{\theta}_0) \sum_{g\epsilon G(i)} \theta_{hj} + A_i^*(\underset{\sim}{\Delta}, \underset{\sim}{\theta}_0) \sum_{g\epsilon G(i)} \theta_{hi}$$

$$= \sum_{\substack{j=1 \\ j\neq i}}^{k} A_j^*(\underset{\sim}{\Delta},\underset{\sim}{\theta}_0)\left[(k-2)! \sum_{\substack{m=1 \\ m\neq i}}^{k} \theta_m\right] + (k-1)!\theta_i A_i^*(\underset{\sim}{\Delta},\underset{\sim}{\theta}_0)$$

$$= (k-2)!(U-\theta_i) \sum_{\substack{j=1 \\ j\neq i}}^{k} A_j^*(\underset{\sim}{\Delta},\underset{\sim}{\theta}_0) + (k-1)!\theta_i A_i^*(\underset{\sim}{\Delta},\underset{\sim}{\theta}_0)$$

$$= (k-2)!(k\theta_i - U)A_i^*(\underset{\sim}{\Delta},\underset{\sim}{\theta}_0)$$

where the first and the last equalities are obtained due to the fact that $\sum_{j=1}^{k} A_j^*(\underset{\sim}{\Delta},\underset{\sim}{\theta}_0) = 0$. This completes the proof of Lemma 3.3.

<u>Theorem 3. 1.</u> Let $\underset{\sim}{\theta} \epsilon \Omega$ be any point in the neighborhood of $\underset{\sim}{\theta}_0 \epsilon \Omega_0$. Let $P_i(\underset{\sim}{\theta}) = E_{\underset{\sim}{\theta}}[\alpha_i(\underset{\sim}{\Delta}(r))]$ be the probability of including population π_i in the selected subset under $\underset{\sim}{\theta}$ by applying an invariant subset selection rule R. Then, under the condition A, for each $i = 1,2,\ldots,k$,

$$P_i(\underset{\sim}{\theta}) \simeq E_{\underset{\sim}{\theta}_0}\left\{\left[1 + \frac{(k\theta_i - U)N!}{k-1} T_i^*(\underset{\sim}{\Delta}(r),\underset{\sim}{\theta}_0)\right]\alpha_i(\underset{\sim}{\Delta}(r))\right\}, \tag{3.7}$$

where

$$T_i^*(\underset{\sim}{\Delta}(r), \underset{\sim}{\theta}_0) = \frac{1}{|L(\underset{\sim}{\Delta}(r))|} \sum_{\underset{\sim}{\Delta}\epsilon L(\underset{\sim}{\Delta}(r))} A_i^*(\underset{\sim}{\Delta}, \underset{\sim}{\theta}_0). \tag{3.8}$$

Proof: It is trivial that under the condition A, $|A_i(\underline{\Delta},\underline{\theta}_0,\underline{\theta})| < \infty$ and $|A_i^*(\underline{\Delta},\underline{\theta}_0)| < \infty$ for all $i = 1, \ldots, k$. It is also clear that $(\theta_j - \theta_0)A_i(\underline{\Delta},\underline{\theta}_0,g\underline{\theta}) = (\theta_j - \theta_0)\, A_i^*(\underline{\Delta},\underline{\theta}_0)$ if $\theta_j = \theta_0$. Thus, we assume that $\theta_j \neq \theta_0$ for each $j = 1, \ldots, k$. Note that $||\underline{\theta}-\underline{\theta}_0|| = ||g\underline{\theta}-\underline{\theta}_0||$ for all $g \in G$. Then, by the assumption and (3.5), we can choose $\varepsilon > 0$ so small that as $||\underline{\theta} - \underline{\theta}_0|| < \varepsilon$, $A_i(\underline{\Delta}, \underline{\theta}_0, g\underline{\theta}) \approx A_i^*(\underline{\Delta}, \underline{\theta}_0)$ for all $g \varepsilon G$ and so $(\theta_{hj} - \theta_0)\, A_i(\underline{\Delta}, \underline{\theta}_0, g\underline{\theta}) \approx (\theta_{hj} - \theta_0)\, A_i^*(\underline{\Delta}, \underline{\theta}_0)$ for all $g \varepsilon G$ where h is the inverse of g. Thus, either $\min_{1\leq i\leq k} |\theta_i - \theta_0| = 0$ or $\min_{1\leq i\leq k} |\theta_i - \theta_0| > 0$, if $||\underline{\theta} - \underline{\theta}_0|| < \varepsilon$, we have

$$\begin{aligned}
&\sum_{g\varepsilon G(i)} \sum_{j=1}^{k} (\theta_{hj} - \theta_0)A_j(\underline{\Delta}, \underline{\theta}_0, g\underline{\theta}) \\
&\approx \sum_{g\varepsilon G(i)} \sum_{j=1}^{k} (\theta_{hj} - \theta_0)A_j^*(\underline{\Delta}, \underline{\theta}_0) \qquad (3.9) \\
&= (k-2)!(k\theta_i - U)A_i^*(\underline{\Delta},\underline{\theta}_0)
\end{aligned}$$

where the last equality is due to Lemma 3.3. Then, by Lemma 3.2 and (3.9), we obtain

$$\begin{aligned}
&P_i(\underline{\theta}) \\
&\approx \sum_{\underline{\Delta}(r)\varepsilon L_r} \left[\frac{(n!)^k}{N!}\,|L(\underline{\Delta}(r))| + \frac{(n!)^k}{(k-1)} \sum_{\underline{\Delta}\varepsilon L(\underline{\Delta}(r))} (k\theta_i - U)A_i^*(\underline{\Delta},\underline{\theta}_0)\right]\alpha_i(\underline{\Delta}(r)) \\
&= E_{\underline{\theta}_0}\left\{\left[1 + \frac{(k\theta_i - U)N!}{k-1}\ \frac{1}{|L(\underline{\Delta}(r))|} \sum_{\underline{\Delta}\varepsilon L(\underline{\Delta}(r))} A_i^*(\underline{\Delta},\underline{\theta}_0)\right]\alpha_i(\underline{\Delta}(r))\right\} \\
&= E_{\underline{\theta}_0}\left\{\left[1 + \frac{(k\theta_i - U)N!}{k-1}\, T_i^*(\underline{\Delta}(r),\underline{\theta}_0)\right]\alpha_i(\underline{\Delta}(r))\right\}.
\end{aligned}$$

This completes the proof of Theorem 3.1.

Now, define subset selection rule R_1 as follows:

$$\alpha_i(\underset{\sim}{\Delta}(r)) = \begin{cases} 1 & \text{if } T_i^*(\underset{\sim}{\Delta}(r), \underset{\sim}{\theta}_0) > c(\underset{\sim}{\theta}_0); \\ \rho(\underset{\sim}{\theta}_0) & \text{if } T_i^*(\underset{\sim}{\Delta}(r), \underset{\sim}{\theta}_0) = c(\underset{\sim}{\theta}_0); \\ 0 & \text{if } T_i^*(\underset{\sim}{\Delta}(r), \underset{\sim}{\theta}_0) < c(\underset{\sim}{\theta}_0); \end{cases} \tag{3.10}$$

where the constants $c(\underset{\sim}{\theta}_0)$ and $\rho(\underset{\sim}{\theta}_0)$, $(0 \leq \rho(\underset{\sim}{\theta}_0) < 1)$, depend on the parameter θ_0, and can be determined by

$$P_{\underset{\sim}{\theta}_0}\{T_1^*(\underset{\sim}{\Delta}(r),\underset{\sim}{\theta}_0) > c(\underset{\sim}{\theta}_0)\} + \rho(\underset{\sim}{\theta}_0)P_{\underset{\sim}{\theta}_0}\{T_1^*(\underset{\sim}{\Delta}(r),\underset{\sim}{\theta}_0) = c(\underset{\sim}{\theta}_0)\} = P^*. \tag{3.11}$$

We then have the following theorem.

<u>Theorem 3. 2.</u> Suppose that the density function $f(x,\theta)$ satisfies the Condition A. Then, the subset selection rule R_1 maximizes $P_{\underset{\sim}{\theta}}(CS|R)$ in a neighborhood of $\underset{\sim}{\theta}_0 \in \Omega_0$, among all invariant subset selection rules, based on the joint type II censored ranks, satisfying $\inf_{\underset{\sim}{\theta} \in \Omega_0} P_{\underset{\sim}{\theta}}(CS|R) = P^*$.

Proof: Without loss of generality, we assume that π_k is the best population. Then by Theorem 3.1 , for any $\underset{\sim}{\theta} \in \Omega_k$ in a neighborhood of $\underset{\sim}{\theta}_0 \in \Omega_0$,

$$P_{\underset{\sim}{\theta}}(CS|R) = P_k(\underset{\sim}{\theta})$$

$$= E_{\underset{\sim}{\theta}_0}\left\{\left[1 + \frac{(k\theta_k-U)N!}{k-1} T_k^*(\underset{\sim}{\Delta}(r),\underset{\sim}{\theta}_0)\right]\alpha_k(\underset{\sim}{\Delta}(r))\right\}.$$

Since $k\theta_k - U = \sum_{j=1}^{k-1} (\theta_k - \theta_j) \geq 0$, then by Neyman-Pearson lemma, we conclude this theorem.

4. Local Monotonicity of the Subset Selection Rule R_1

Let R be a subset selection rule and $P_i(\underset{\sim}{\theta})$ be the associated probability of including population π_i in the selected subset for each $i = 1,\ldots, k$, when $\underset{\sim}{\theta}$ is the true parameter.

Note that by definition of $P_i(\underset{\sim}{\theta})$,

$$\begin{aligned} P_i(\underset{\sim}{\theta}) &= E_{\underset{\sim}{\theta}}[\alpha_i(\underset{\sim}{\Delta}(r))] \\ &= \sum_{\underset{\sim}{\Delta}(r)\in L_r} \left[\sum_{\underset{\sim}{\Delta}\in L(\underset{\sim}{\Delta}(r))} P_{\underset{\sim}{\theta}}(\underset{\sim}{\Delta})\right] \alpha_i(\underset{\sim}{\Delta}(r)) \end{aligned} \tag{4.1}$$

where $P_{\underset{\sim}{\theta}}(\underset{\sim}{\Delta})$ is defined in (3.1).

Let $\underset{\sim}{\theta}^* = (\theta_1^*,\ldots,\theta_k^*) \in \Omega$. Under Condition A,

$$\left.\frac{\partial P_{\underset{\sim}{\theta}}(\underset{\sim}{\Delta})}{\partial\theta_j}\right|_{\underset{\sim}{\theta}=\underset{\sim}{\theta}^*}$$ exists for each $j = 1,\ldots,k$ and

$$\lim_{||\underset{\sim}{\theta}^*-\underset{\sim}{\theta}_0||\to 0} \left.\frac{\partial P_{\underset{\sim}{\theta}}(\underset{\sim}{\Delta})}{\partial\theta_j}\right|_{\underset{\sim}{\theta}=\underset{\sim}{\theta}^*} = \left.\frac{\partial P_{\underset{\sim}{\theta}}(\underset{\sim}{\Delta})}{\partial\theta_j}\right|_{\underset{\sim}{\theta}=\underset{\sim}{\theta}_0} = \sum_{\substack{m=1 \\ \Delta_m=j}}^{N} B_m(\underset{\sim}{\theta}_0)\times(n!)^k = A_j^*(\underset{\sim}{\Delta},\underset{\sim}{\theta}_0)(n!)^k,$$

where $B_m(\underset{\sim}{\theta}_0)$, $A_j^*(\underset{\sim}{\Delta},\underset{\sim}{\theta}_0)$ are defined in (3.6) and (3.5), respectively.

Therefore, we have

$$\begin{aligned} \lim_{||\underset{\sim}{\theta}^*-\underset{\sim}{\theta}_0||\to 0} \left.\frac{\partial P_i(\underset{\sim}{\theta})}{\partial\theta_i}\right|_{\underset{\sim}{\theta}=\underset{\sim}{\theta}^*} &= \left.\frac{\partial P_i(\underset{\sim}{\theta})}{\partial\theta_i}\right|_{\underset{\sim}{\theta}=\underset{\sim}{\theta}_0} \\ &= (n!)^k \sum_{\underset{\sim}{\Delta}(r)\in L_r} \left[\sum_{\underset{\sim}{\Delta}\in L(\underset{\sim}{\Delta}(r))} A_i^*(\underset{\sim}{\Delta},\underset{\sim}{\theta}_0)\right]\alpha_i(\underset{\sim}{\Delta}(r)), \end{aligned} \tag{4.2}$$

and

$$\lim_{||\underset{\sim}{\theta}^*-\underset{\sim}{\theta}_0||\to 0} \left.\frac{\partial P_i(\underset{\sim}{\theta})}{\partial \theta_j}\right|_{\underset{\sim}{\theta}=\underset{\sim}{\theta}^*} = \left.\frac{\partial P_i(\underset{\sim}{\theta})}{\partial \theta_j}\right|_{\underset{\sim}{\theta}=\underset{\sim}{\theta}_0}$$

$$= (n!)^k \sum_{\underset{\sim}{\Delta}(r)\in L_r} \left[\sum_{\Delta\in L(\underset{\sim}{\Delta}(r))} A_j^*(\underset{\sim}{\Delta},\underset{\sim}{\theta}_0)\right]\alpha_i(\underset{\sim}{\Delta}(r)) \qquad (4.3)$$

$$\forall \quad j \neq i.$$

<u>Definition 4.1.</u> A subset selection rule R is locally strongly monotone at point $\underset{\sim}{\theta}_0 \in \Omega_0$ if for each $i = 1, \ldots, k$, $\left.\frac{\partial P_i(\underset{\sim}{\theta})}{\partial \theta_i}\right|_{\underset{\sim}{\theta}=\underset{\sim}{\theta}_0} \geq 0$ and $\left.\frac{\partial P_i(\underset{\sim}{\theta})}{\partial \theta_j}\right|_{\underset{\sim}{\theta}=\underset{\sim}{\theta}_0} \leq 0$ for all $j \neq i$.

The following lemmas are needed for deriving the locally strong monotonicity of the subset selection rule R_1.

<u>Lemma 4.1</u> Let $g\in G$ and $\bar{g}\in\bar{G}$ and $\bar{\bar{g}}\in\bar{\bar{G}}$ are that induced from g. Then, for any $\underset{\sim}{\Delta} \in L$, $\underset{\sim}{\Delta}(r)\in L_r$, we have

$$A_{gi}^*(\bar{g}\underset{\sim}{\Delta}, \underset{\sim}{\theta}_0) = A_i^*(\underset{\sim}{\Delta}, \underset{\sim}{\theta}_0)$$

for each $i = 1, \ldots, k$.

Proof: From (3.5), we have

$$A_i^*(\underset{\sim}{\Delta}, \underset{\sim}{\theta}_0) = \sum_{\substack{j=1\\ \Delta_j=i}}^{N} B_j(\underset{\sim}{\theta}_0) = \sum_{\substack{j=1\\ g\Delta_j=gi}}^{N} B_j(\underset{\sim}{\theta}_0)$$

$$= \sum_{\substack{j=1\\ (\bar{g}\underset{\sim}{\Delta})_j=gi}}^{N} B_j(\underset{\sim}{\theta}_0) = A_{gi}^*(\bar{g}\underset{\sim}{\Delta}, \underset{\sim}{\theta}_0) .$$

Now, we see that for each $i = 1, \ldots, k$, $A_i^*(\underset{\sim}{\Delta}, \underset{\sim}{\theta}_0)$ depends on $\underset{\sim}{\Delta}$ only through whether $\Delta_j = i$ or not for each $j = 1, \ldots, N$, and when $\Delta_j \neq i$, then $A_i^*(\underset{\sim}{\Delta}, \underset{\sim}{\theta}_0)$ is independent of the value of Δ_j. Similarly, $T_i^*(\underset{\sim}{\Delta}(r), \underset{\sim}{\theta}_0)$ depends on $\underset{\sim}{\Delta}(r)$ only through whether $\Delta_j = i$ or not for each $j = 1, \ldots, r$, and when $\Delta_j \neq i$, then $T_i^*(\underset{\sim}{\Delta}(r), \underset{\sim}{\theta}_0)$ is independent of the value of Δ_j. Thus, for the subset selection rule R_1, $\alpha_i(\underset{\sim}{\Delta}(r))$ depends on $\underset{\sim}{\Delta}(r)$ only through whether $\Delta_j = i$ or not for each $j = 1, \ldots, r$.

Let $g \epsilon G(i)$. Since g does not change the position of index i, therefore, for each $\underset{\sim}{\Delta}(r) \, \epsilon \mathcal{L}_r$, $\alpha_i(\bar{\bar{g}}\underset{\sim}{\Delta}(r)) = \alpha_i(\underset{\sim}{\Delta}(r))$ where $\bar{\bar{g}} \epsilon \bar{\bar{G}}$ is induced from g. Now, according to the value of $\alpha_i(\underset{\sim}{\Delta}(r))$, the set $\mathcal{L}_r$ can be partitioned into three classes, say,

$\mathcal{L}_r = \mathcal{L}_r^i(0) \cup \mathcal{L}_r^i(1) \cup \mathcal{L}_r^i(\rho(\underset{\sim}{\theta}_0))$ where $\mathcal{L}_r^i(\beta) = \{\underset{\sim}{\Delta}(r) \, \epsilon \mathcal{L}_r | \alpha_i(\underset{\sim}{\Delta}(r)) = \beta\}$ for $\beta = 0, 1$ or $\rho(\underset{\sim}{\theta}_0)$.

<u>Lemma 4.2.</u> Let $g \epsilon G(i)$ and $\bar{\bar{g}} \epsilon \bar{\bar{G}}$ be the one induced from g. Then

$$\bar{\bar{g}}(\mathcal{L}_r^i(\beta)) = \mathcal{L}_r^i(\beta) \quad \text{for each } \beta = 0, 1 \text{ or } \rho(\underset{\sim}{\theta}_0).$$

Proof: For each β, let $\underset{\sim}{\Delta}(r) \epsilon \mathcal{L}_r^i(\beta)$. Then $\alpha_i(\underset{\sim}{\Delta}(r)) = \beta$ and so $\alpha_i(\bar{\bar{g}}\underset{\sim}{\Delta}(r)) = \beta$ since $g \epsilon G(i)$. Therefore $\bar{\bar{g}}\underset{\sim}{\Delta}(r) \epsilon \mathcal{L}_r^i(\beta)$. That is, $\bar{\bar{g}}(\mathcal{L}_r^i(\beta)) \subseteq \mathcal{L}_r^i(\beta)$. Also, $\bar{\bar{g}}\mathcal{L}_r = \mathcal{L}_r$. Thus, if $\bar{\bar{g}}\mathcal{L}_r^i(\beta) \subsetneq \mathcal{L}_r^i(\beta)$ for some β, we then have $\bar{\bar{g}}\mathcal{L}_r \subsetneq \mathcal{L}_r$ which is a contradiction. Therefore, $\bar{\bar{g}}(\mathcal{L}_r^i(\beta)) = \mathcal{L}_r^i(\beta)$ for each $\beta = 0, 1$ or $\rho(\underset{\sim}{\theta}_0)$.

Lemma 4.3. For each fixed i and $m \neq i$, $j \neq i$ and $m \neq j$, we have

$$\sum_{\underset{\sim}{\Delta}(r)\in L_r^i(\beta)} \left[\sum_{\underset{\sim}{\Delta}\in L(\underset{\sim}{\Delta}(r))} A_j^*(\underset{\sim}{\Delta}, \underset{\sim}{\theta}_0)\right]\alpha_i(\underset{\sim}{\Delta}(r))$$

$$= \sum_{\underset{\sim}{\Delta}(r)\in L_r^i(\beta)} \left[\sum_{\underset{\sim}{\Delta}\in L(\underset{\sim}{\Delta}(r))} A_m^*(\underset{\sim}{\Delta}, \underset{\sim}{\theta}_0)\right]\alpha_i(\underset{\sim}{\Delta}(r))$$

for each $\beta = 0, 1$ or $\rho(\underset{\sim}{\theta}_0)$.

Proof: Let $g\epsilon G(i)$ and satisfies that $gm = j$, $gj = m$. Then,

$$\sum_{\underset{\sim}{\Delta}(r)\in L_r^i(\beta)} \left[\sum_{\underset{\sim}{\Delta}\in L(\underset{\sim}{\Delta}(r))} A_j^*(\underset{\sim}{\Delta}, \underset{\sim}{\theta}_0)\right]\alpha_i(\underset{\sim}{\Delta}(r))$$

$$= \beta \sum_{\underset{\sim}{\Delta}(r)\in L_r^i(\beta)} \left[\sum_{\underset{\sim}{\Delta}\in L(\underset{\sim}{\Delta}(r))} A_{gj}^*(\bar{g}\underset{\sim}{\Delta}, \underset{\sim}{\theta}_0)\right] \quad \text{(by Lemma 4.1)}$$

$$= \beta \sum_{\underset{\sim}{\Delta}(r)\in L_r^i(\beta)} \left[\sum_{\underset{\sim}{\Delta}\in L(\underset{\sim}{\Delta}(r))} A_m^*(\bar{g}\underset{\sim}{\Delta}, \underset{\sim}{\theta}_0)\right]$$

$$= \beta \sum_{\underset{\sim}{\Delta}(r)\in L_r^i(\beta)} \left[\sum_{\underset{\sim}{\Delta}\in L(\bar{g}\underset{\sim}{\Delta}(r))} A_m^*(\underset{\sim}{\Delta}, \underset{\sim}{\theta}_0)\right]$$

$$= \beta \sum_{\underset{\sim}{\Delta}(r)\in \bar{g} L_r^i(\beta)} \left[\sum_{\underset{\sim}{\Delta}\in L(\underset{\sim}{\Delta}(r))} A_m^*(\underset{\sim}{\Delta}, \underset{\sim}{\theta}_0)\right]$$

$$= \beta \sum_{\underset{\sim}{\Delta}(r)\in L_r^i(\beta)} \left[\sum_{\underset{\sim}{\Delta}\in L(\underset{\sim}{\Delta}(r))} A_m^*(\underset{\sim}{\Delta}, \underset{\sim}{\theta}_0)\right] \quad \text{(by Lemma 4.2)}$$

This completes the proof of Lemma 4.3.

The following corollary is a direct application of Lemma 4.3.

Corollary 4.1. For each fixed i and $m \neq i$, $j \neq i$, we have

$$\left.\frac{\partial P_i(\underset{\sim}{\theta})}{\partial \theta_j}\right|_{\underset{\sim}{\theta}=\underset{\sim}{\theta}_0} = \left.\frac{\partial P_i(\underset{\sim}{\theta})}{\partial \theta_m}\right|_{\underset{\sim}{\theta}=\underset{\sim}{\theta}_0}.$$

Theorem 4.1. Suppose that the density function $f(x,\theta)$ satisfies the Condition A. Then, the subset selection rule R_1 is locally strongly monotone at each $\underset{\sim}{\theta}_0 \in \Omega_0$.

Proof: By Corollary 4.1, for each $m \neq i$, we have

$$\begin{aligned}
\left.\frac{\partial P_i(\underset{\sim}{\theta})}{\partial \theta_m}\right|_{\underset{\sim}{\theta}=\underset{\sim}{\theta}_0} &= \frac{1}{k-1} \sum_{\substack{j=1 \\ j\neq i}}^{k} \left.\frac{\partial P_i(\underset{\sim}{\theta})}{\partial \theta_j}\right|_{\underset{\sim}{\theta}=\underset{\sim}{\theta}_0} \\
&= \frac{(n!)^k}{k-1} \sum_{\substack{j=1 \\ j\neq i}}^{k} \sum_{\underset{\sim}{\Delta}(r)\in L_r} \left[\sum_{\underset{\sim}{\Delta}\in L(\underset{\sim}{\Delta}(r))} A_j^*(\underset{\sim}{\Delta},\underset{\sim}{\theta}_0)\right] \alpha_i(\underset{\sim}{\Delta}(r)) \\
&= \frac{(n!)^k}{k-1} \sum_{\underset{\sim}{\Delta}(r)\in L_r} \left[\sum_{\underset{\sim}{\Delta}\in L(\underset{\sim}{\Delta}(r))} \sum_{\substack{j=1 \\ j\neq i}}^{k} A_j^*(\underset{\sim}{\Delta},\underset{\sim}{\theta}_0)\right] \alpha_i(\underset{\sim}{\Delta}(r)) \\
&= -\frac{(n!)^k}{k-1} \sum_{\underset{\sim}{\Delta}(r)\in L_r} \left[\sum_{\underset{\sim}{\Delta}\in L(\underset{\sim}{\Delta}(r))} A_i^*(\underset{\sim}{\Delta},\underset{\sim}{\theta}_0)\right] \alpha_i(\underset{\sim}{\Delta}(r)) \\
&= -\frac{1}{k-1} \left.\frac{\partial}{\partial \theta_i} P_i(\underset{\sim}{\theta})\right|_{\underset{\sim}{\theta}=\underset{\sim}{\theta}_0},
\end{aligned}$$

where the last second equality is due to Lemma 3.1.

Therefore, it suffices to prove that $\frac{\partial}{\partial\theta_i} P_i(\underset{\sim}{\theta})\Big|_{\underset{\sim}{\theta}=\underset{\sim}{\theta}_0} \geq 0$ for each $\underset{\sim}{\theta}_0 \in \Omega_0$. Now,

$$\sum_{\underset{\sim}{\Delta}(r)\in L_r} |L(\underset{\sim}{\Delta}(r))| T_i^*(\underset{\sim}{\Delta}(r), \underset{\sim}{\theta}_0)$$

$$= \sum_{\underset{\sim}{\Delta}(r)\in L_r} \sum_{\Delta\in L(\underset{\sim}{\Delta}(r))} A_i^*(\underset{\sim}{\Delta}, \underset{\sim}{\theta}_0)$$

$$= \sum_{\underset{\sim}{\Delta}\in L} A_i^*(\underset{\sim}{\Delta}, \underset{\sim}{\theta}_0) \tag{4.4}$$

$$= \sum_{\underset{\sim}{\Delta}\in L} \sum_{j=1}^{N} B_j(\underset{\sim}{\theta}_0) I_{\{i\}}(\Delta_j)$$

$$= \sum_{j=1}^{N} \left[B_j(\underset{\sim}{\theta}_0) \sum_{\underset{\sim}{\Delta}\in L} I_{\{i\}}(\Delta_j)\right]$$

$$= \frac{(N-1)!}{(n!)^{k-1}(n-1)!} \sum_{j=1}^{N} B_j(\underset{\sim}{\theta}_0)$$

$$= 0,$$

since $\sum_{j=1}^{N} B_j(\underset{\sim}{\theta}_0) = 0$ which is due to Lemma 3.1 under Condition A.

Then, by (3.10) and (4.4), we see that

$$\sum_{\underset{\sim}{\Delta}(r)\in L_r} \left[\sum_{\underset{\sim}{\Delta}\in L(\underset{\sim}{\Delta}(r))} A_i^*(\underset{\sim}{\Delta}, \underset{\sim}{\theta}_0)\right] a_i(\underset{\sim}{\Delta}(r)) \geq 0$$

Therefore $\frac{\partial P_i(\underset{\sim}{\theta})}{\partial\theta_i}\Big|_{\underset{\sim}{\theta}=\underset{\sim}{\theta}_0} \geq 0$. Hence, the subset selection rule R_1 is locally strongly monotone at each $\underset{\sim}{\theta}_0 \in \Omega_0$.

5. Remarks.

(1) Note that when $r = N$, that is, in the complete rank configuration case, this locally optimal subset selection rule R_1 turns out to be the one studied by Gupta, Huang and Nagel (1979).

(2) This locally optimal subset selection rule R_1 is based on the weighted rank sum $B_j(\underset{\sim}{\theta}_0) = \frac{1}{(j-1)!(N-j)!} \int_0^1 u^{j-1} (1-u)^{N-j} \phi(u, f, \theta_0) du$ where $\phi(u, f, \theta) = \dot{f}(F^{-1}(u, \theta), \theta)/f(F^{-1}(u, \theta), \theta)$. In general, $\phi(u, f, \theta)$ depends on θ. However, it is independent of θ if θ is a location parameter (see Gupta, Huang and Nagel (1979)). In this situation, the value $B_j(\underset{\sim}{\theta}_0)$ is independent of $\underset{\sim}{\theta}_0$. Therefore, the two constants $c(\underset{\sim}{\theta}_0)$ and $\rho(\underset{\sim}{\theta}_0)$, which are used to determine the rule R_1, are also independent of $\underset{\sim}{\theta}_0$ for each fixed p^* value.

(3) Suppose that $\theta > 0$ is a scale parameter, that is, $f(x, \theta) = \theta h(\theta(x - \mu))$ for some function $h(\cdot)$. Let $\theta_1, \theta_2 > 0$ such that $\theta_2 = \beta\theta_1$. Then, $\phi(u, f, \theta_2) = \frac{1}{\beta} \phi(u, f, \theta_1)$. Therefore, $B_j(\underset{\sim}{\theta}_2) = \frac{1}{\beta} B_j(\underset{\sim}{\theta}_1)$ for each $j = 1, \ldots, N$, where $\underset{\sim}{\theta}_i = (\theta_i, \ldots, \theta_i) \in \Omega_0$, $i = 1, 2$. In this situation, for each fixed p^* value, we have $c(\underset{\sim}{\theta}_2) = \frac{1}{\beta} c(\underset{\sim}{\theta}_1)$ and $\rho(\underset{\sim}{\theta}_2) = \rho(\underset{\sim}{\theta}_1)$.

Huang and Panchapakesan (1982) also derived a subset selection rule, say R_{HP}, based on the complete rank configurations, which can be represented as follows:

$$a_i(\underset{\sim}{A}) = \begin{cases} 1 & \text{if } A_i^*(\underset{\sim}{A}, \underset{\sim}{\theta}_0) > V(\theta_0) + D \\ \rho & \text{if } A_i^*(\underset{\sim}{A}, \underset{\sim}{\theta}_0) = V(\theta_0) + D \\ 0 & \text{if } A_i^*(\underset{\sim}{A}, \underset{\sim}{\theta}_0) < V(\theta_0) + D \end{cases} \tag{5.1}$$

where D and $\rho(0 \le \rho < 1)$ are chosen so that

$$P_{\underset{\sim}{\theta}_0}\{A_i^*(\underset{\sim}{\Delta}, \underset{\sim}{\theta}_0) > V(\theta_0) + D\} + \rho P_{\underset{\sim}{\theta}_0}\{A_i^*(\underset{\sim}{\Delta}, \underset{\sim}{\theta}_0) = V(\theta_0) + D\} = P^* \quad (5.2)$$

$$\text{and } V(\theta_0) + D > 0.$$

The rule R_{HP} is always locally strongly monotone provided the constants D and ρ satisfying (5.2) exist. However, as pointed out by themselves, it is possible that the D and ρ satisfying (5.2) may not exist. In such a case, the rule R_{HP} selects the empty subset. The following example indicates that the rule R_{HP} always selects the empty subset when $p^* > \frac{1}{2}$.

<u>Example</u> Let k (≥ 2) and n be positive integers and let $N = kn$. Let $f(x,\theta)$ be the logistic density $f(x,\theta) = e^{-(x-\theta)}/[1 + e^{-(x-\theta)}]^2$, $-\infty < x < \infty$, $-\infty < \theta < \infty$. It is clear that $f(x,\theta)$ satisfies the Condition A. Then by Lemma 3.1 , $V(\theta_0) = 0$ for all $\underset{\sim}{\theta}_0 \varepsilon \Omega_0$. Also, $\phi(u, f, \theta) = 2u - 1$, which leads to equally spaced scores and

$$B_j(\underset{\sim}{\theta}_0) = \frac{2j}{(N+1)!} - \frac{1}{N!} . \quad (5.3)$$

Note that $B_j(\underset{\sim}{\theta}_0) + B_{N+1-j}(\underset{\sim}{\theta}_0) = 0$ for each $j = 1, \ldots, N$. Therefore, for each $\underset{\sim}{\Delta} = (\Delta_1, \ldots, \Delta_N) \varepsilon \mathcal{L}$, let $\underset{\sim}{\Delta}^1 = (\Delta_1^1, \ldots, \Delta_N^1)$ where $\Delta_j^1 = \Delta_{N+1-j}$ for each $j = 1, \ldots, N$. Then $\underset{\sim}{\Delta}^1 \varepsilon \mathcal{L}$. By (5.3) and the definition of $A_i^*(\underset{\sim}{\Delta}, \underset{\sim}{\theta}_0)$, we have $A_i^*(\underset{\sim}{\Delta}, \underset{\sim}{\theta}_0) + A_i^*(\underset{\sim}{\Delta}^1, \underset{\sim}{\theta}_0) = 0$ for all $\underset{\sim}{\Delta} \varepsilon \mathcal{L}$, $\underset{\sim}{\theta}_0 \varepsilon \Omega_0$, which implies that $P_{\underset{\sim}{\theta}_0}\{A_i^*(\underset{\sim}{\Delta}, \underset{\sim}{\theta}_0) > 0\} \le \frac{1}{2}$

for all $\underset{\sim}{\theta}_0 \in \Omega_0$. Hence, for $p^* > \frac{1}{2}$, there exist no D and ρ $(0 \leq \rho < 1)$ such that (5.2) is satisfied.

However, for the subset selection rule R_1, the corresponding two constants $c(\underset{\sim}{\theta}_0)$ and $\rho(\underset{\sim}{\theta}_0)$ always exist when $p^* \in (\frac{1}{k}, 1)$, and the rule R_1 is always locally strongly monotone which is guaranteed by Theorem 4.1.

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This paper deals with the derivation of subset selection rules which satisfy the basic P^*-condition and which locally maximize the probability of a correct selection among all invariant subset selection rules based on the ranks under the joing type II censoring. Following the earlier setup of Gupta, Huang and Nagel (1979), a locally optimal subset selection rule R_1 is derived. The property of local monotonicity related to the rule R_1 is discussed.

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